Minimum energy states in Hall MHD relaxation theory

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Outline

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- Statement of the problem
- Hall MHD relaxed states:

  - Zero cross helicity

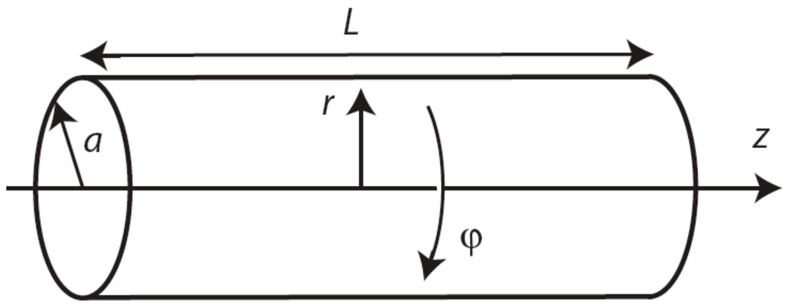
  - Nonzero cross helicity
- Summary

Introduction

- Many magnetized plasma systems exhibit the phenomenon of relaxation (self-organization): these systems tend toward preferred simple configurations
- During relaxation, invariants (quantities conserved in ideal system) decay at different rates due to dissipation.
- Taylor’s conjecture (1974): relaxed states are the states of lowest energy subject to constraints imposed by slowly decaying invariants
- Our goal: determine these states for the cylindrical plasma pinch in the frame of Hall MHD

Statement of the problem

- Geometry



- Conducting walls
- Axial magnetic flux: Φ_0
- Unit of field: $B_0=\Phi_0/(\pi a^2)$
- Unit of length: radius a

- Ideal Hall MHD

$$\nabla \cdot \mathbf{v} = 0$$
$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = (\nabla \times \mathbf{b}) \times \mathbf{b}$$
$$\dot{\mathbf{b}} = \nabla \times (\mathbf{v} \times \mathbf{b} - \delta_i (\nabla \times \mathbf{b}) \times \mathbf{b})$$
$$\nabla \cdot \mathbf{b} = 0$$

- Normalization:

$$\mathbf{V} = V_A \mathbf{v}, \quad \mathbf{B} = B_0 \mathbf{b}, \quad t = \frac{a}{V_A}\tau, \quad P = \rho V_A^2 p.$$

- Hall parameter:

$$\delta_i = \frac{d_i}{a} = \frac{c}{a}\sqrt{\frac{m_i}{4\pi n e^2}}$$

- Transition to MHD: $\delta_i \to 0$

Invariants in ideal Hall MHD

- Energy: $$E = \frac{1}{2}\int (\mathbf{v}^2 + \mathbf{b}^2) d^3\mathbf{r}$$
- Magnetic helicity: $$h_1 = \int \mathbf{A} \cdot \mathbf{b}\, d^3\mathbf{r} = K, \quad \mathbf{b} = \nabla \times \mathbf{A}$$
- Cross helicity: $$h_2 = \int \mathbf{v} \cdot (\mathbf{b} + \frac{1}{2}\delta_i \nabla \times \mathbf{v}) d^3\mathbf{r} = M$$
- Magnetic flux: $$h_3 = \int \mathbf{b} \cdot \mathbf{e}_z dS = \pi$$
- Axial momentum: $$h_4 = \int \mathbf{v} \cdot \mathbf{e}_z d^3\mathbf{r} = 0$$
- Angular momentum: $$h_5 = \int r\mathbf{v} \cdot \mathbf{e}_\varphi d^3\mathbf{r} = 0$$
- Relaxed state: **b** and **v** that minimize *E* and satisfy the constraints $h_1 = K$, $h_2 = M$, $h_3 = \pi$, $h_4 = 0$, $h_5 = 0$

Hall MHD relaxed states

- Relaxed states ($\mu_{1\text{-}5}$ are Lagrange multipliers)

$$\begin{cases} \mathbf{v}_0 = \delta_i \nabla \times \mathbf{b}_0 + (2\delta_i \mu_1 - \mu_2)\mathbf{b}_0 - \mu_4 \mathbf{e}_z - \mu_5 r \mathbf{e}_\varphi, \\ \delta_i \mu_2 \nabla \times \nabla \times \mathbf{b}_0 + (1 - \mu_2^2 + 2\delta_i \mu_1 \mu_2) \nabla \times \mathbf{b}_0 + 2\mu_1 \mathbf{b}_0 = 2\mu_2 \mu_5 \mathbf{e}_z, \\ h_1 = K, \quad h_2 = M, \quad h_3 = \pi, \quad h_4 = 0, \quad h_5 = 0 \end{cases}$$

- General solution has a form (Mahajan, Yoshida, 1998):

$$\mathbf{b}_0 = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \frac{\mu_2 \mu_5}{\mu_1} \mathbf{e}_z, \quad \nabla \times \mathbf{b}_{1,2} = \lambda_{1,2} \mathbf{b}_{1,2},$$

$$\text{where} \quad \delta_i \mu_2 \lambda^2 + (1 - \mu_2^2 + 2\delta_i \mu_1 \mu_2)\lambda + 2\mu_1 = 0$$

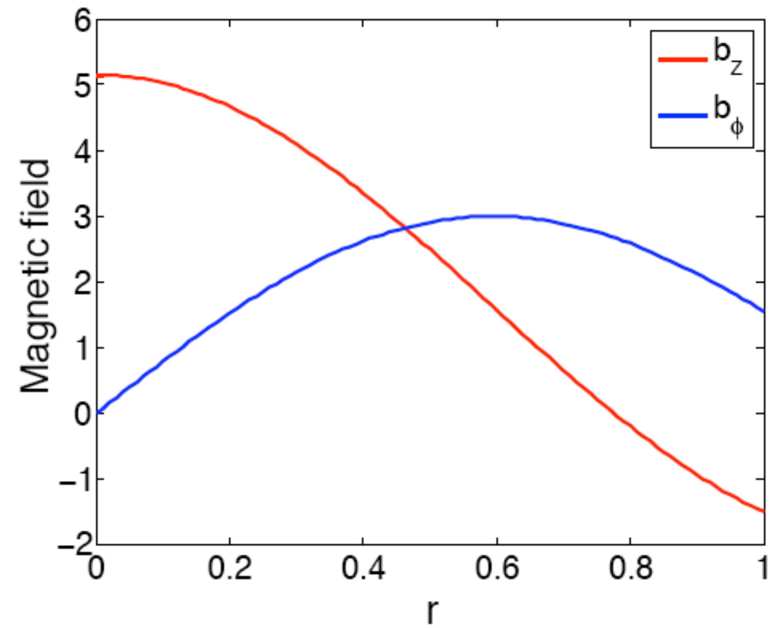
- Two different spatial scales: λ_1 and λ_2
- Relaxed states are completely described by three parameters: magnetic helicity *K*, cross helicity *M* and Hall parameter δ_i
- Consider only axisymmetric states, then

$$\mathbf{b}_{1,2} = J_1(\lambda_{1,2} r)\mathbf{e}_\varphi + J_0(\lambda_{1,2} r)\mathbf{e}_z$$

where J_1 and J_0 are Bessel functions

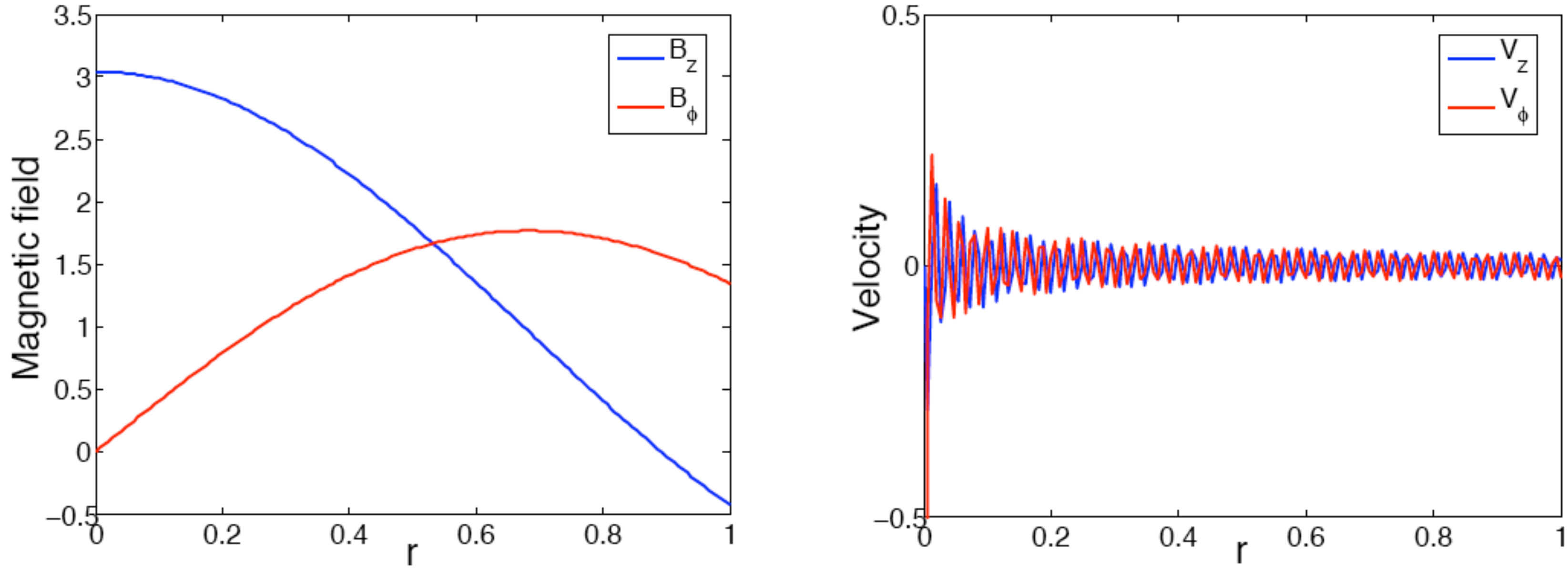
Zero cross helicity, M=0

- Assume $\int \mathbf{v}\cdot(\mathbf{b}+\frac{1}{2}\delta_i\nabla\times\mathbf{v})d^3\mathbf{r}=0$
- One possible solution (independent of Hall parameter δ_i) is Taylor state with $\nabla\times\mathbf{b}_0=\lambda\mathbf{b}_0$, $\mathbf{v}_0=0$
- Taylor state is indeed a minimum energy state in both standard and Hall MHD when M=0
- Magnetic field for K=2



Nonzero cross helicity, $M \neq 0$

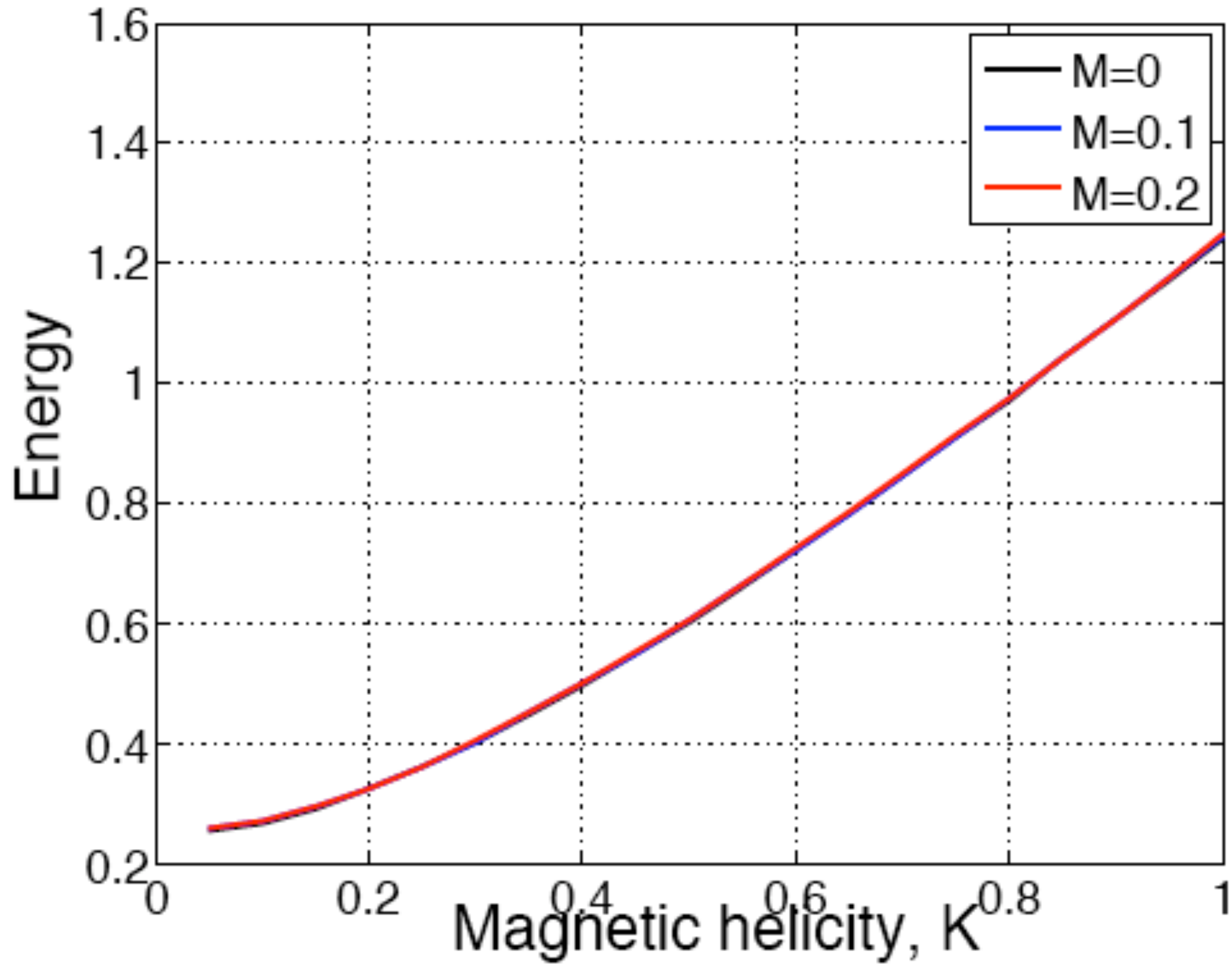
- Magnetic field and velocity for K=0.9, M=0.2, δ_i=0.2



- Different spatial scales play role
- Only second term is important in cross helicity

$$\int (\mathbf{v} \cdot \mathbf{b} + \frac{1}{2} \delta_i \mathbf{v} \cdot \nabla \times \mathbf{v}) d^3\mathbf{r} = M$$

Energy vs. magnetic helicity, δ_i=0.2

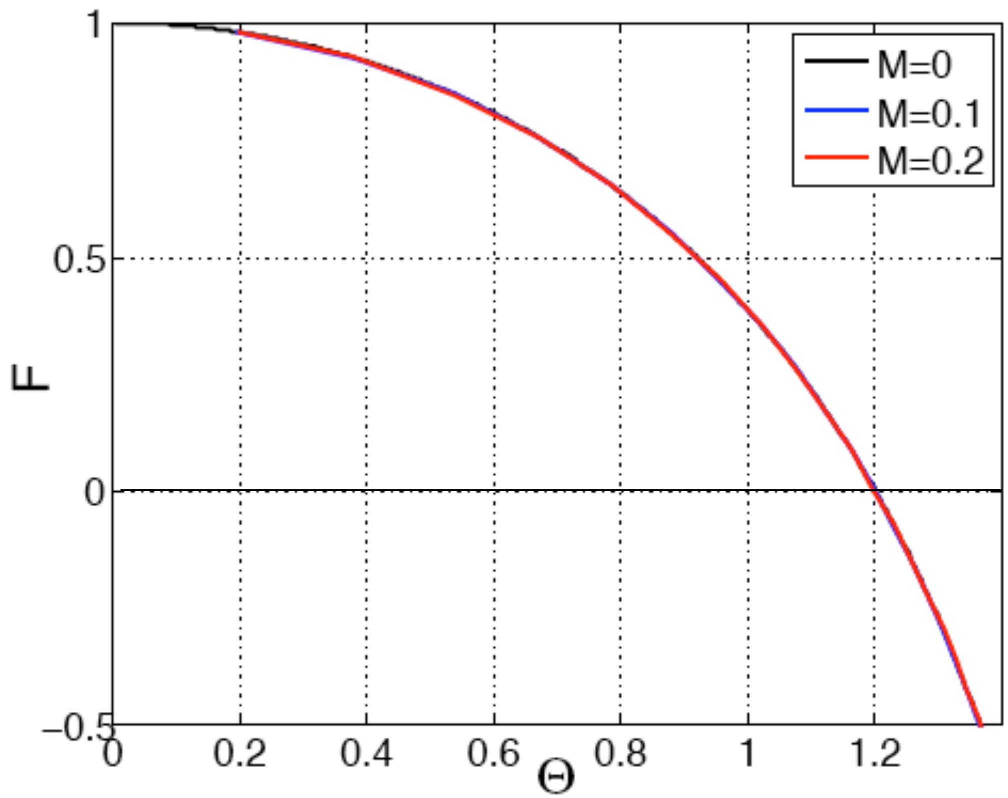


- Not a big difference from Taylor state!

Reversal vs. pinch

- Reversal parameter: $F = \dfrac{B_z(a)}{\langle B_z \rangle}$

- Pinch parameter: $\Theta = \dfrac{B_\varphi(a)}{\langle B_z \rangle}$

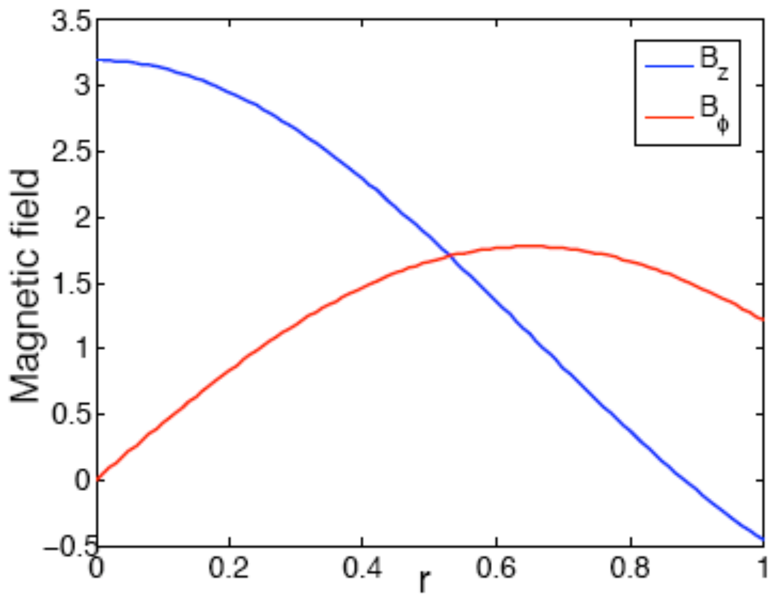


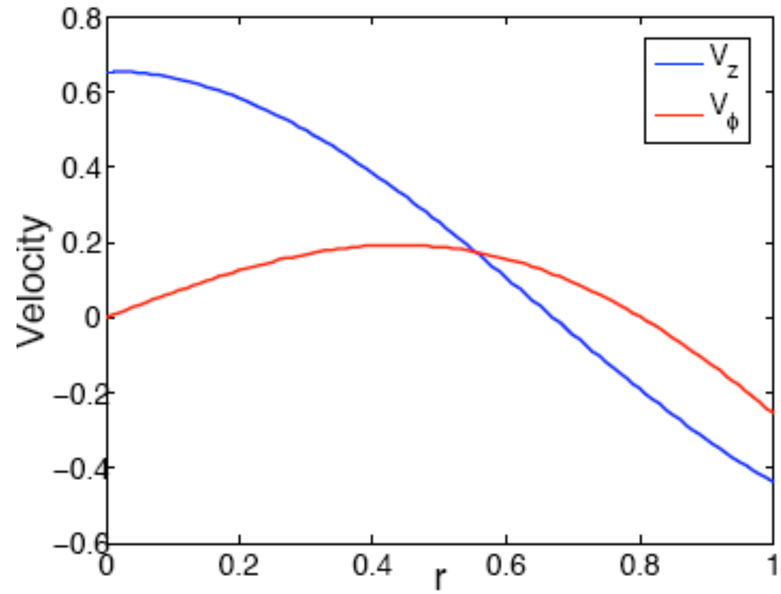
Summary

- Axisymmetric relaxed states of cylindrical plasma pinch are considered in the frame of Hall MHD
- The relaxed states are characterized by three parameters: magnetic helicity K, cross helicity M and Hall parameter δ_i
- For M=0 the minimum energy state is standard Taylor state
- For $M \neq 0$ the minimum energy state is very close to Taylor state

MHD relaxed state

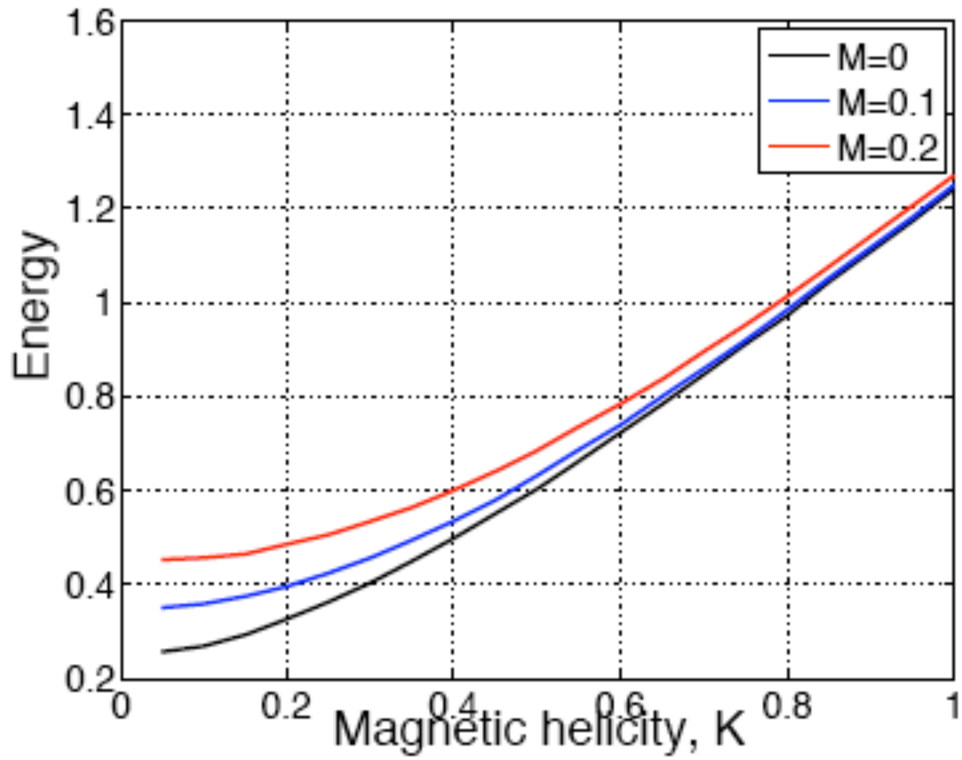
- Magnetic field and velocity for *K*=0.9, *M*=0.2





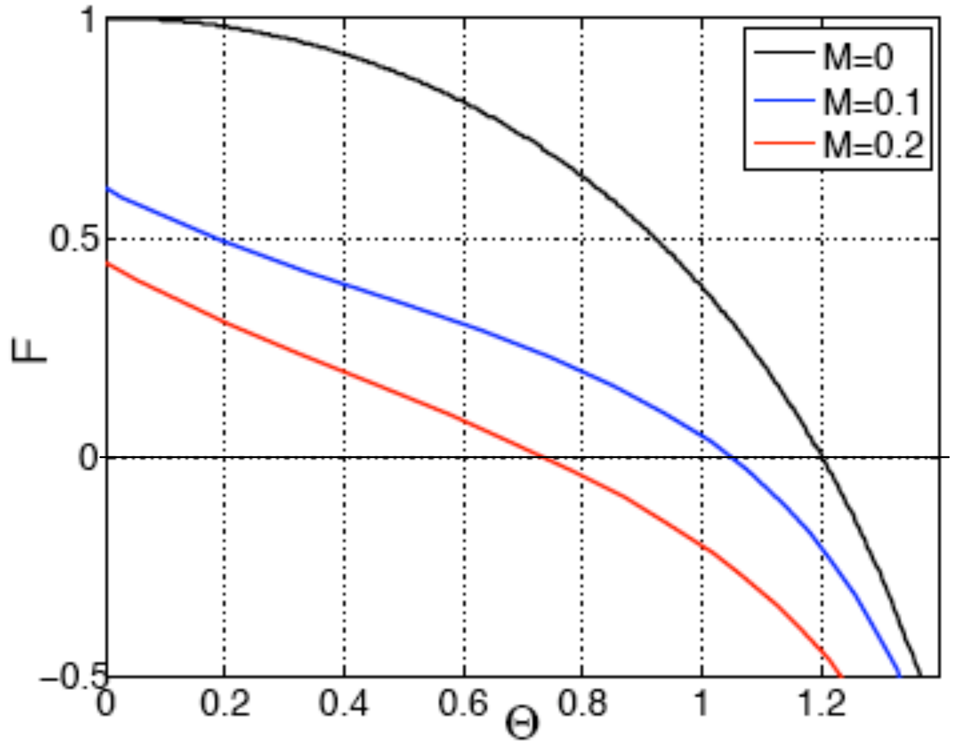
- One spatial scale
- Relatively large flows

Energy vs. magnetic helicity in MHD



- Energy grows with both magnetic and cross helicities

Reversal vs. pinch in MHD



- Field reversal happens at lower pinch parameter for nonzero cross-helicity