Dynamo in helical turbulence

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Introduction and Outline

- Turbulent dynamo converts kinetic energy of a turbulent plasma flow into magnetic energy by random stretching of magnetic field lines frozen into the plasma. Dynamo generates and/or redistributes fields in cosmic plasmas. Observed magnetic fields are often correlated at scales larger than the scales of plasma velocities. Such fields can be explained if there is nonzero kinetic helicity, $H = \int \mathbf{v} \cdot (\nabla \times \mathbf{v})\, d^3x \neq 0$.
- I will present a full numerical solution for the Kazantsev-Kraichnan model of the kinematic turbulent dynamo with helicity.
- I will show magnetic field structure and grow rates for Kolmogorov turbulence (with Reynolds number Re >> 1, magnetic Reynolds number Rm >> 1, and small/large Prandtl number).
- The results indicate a limited applicability of the conventional alpha-model of a large-scale dynamo action, $\partial_t \overline{\mathbf{B}} = \alpha \nabla \times \overline{\mathbf{B}} + \beta \nabla^2 \overline{\mathbf{B}}$.

Kinematic Kazantsev-Kraichnan model

Consider induction equation

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$

Homogeneous isotropic turbulent velocities are prescribed as being a Gaussian white noise:

$$\langle v^i(\mathbf{x},t) v^j(\mathbf{x}',t') \rangle = \kappa^{ij}(|\mathbf{x}-\mathbf{x}'|)\delta(t-t'), \qquad \langle \mathbf{v} \rangle = 0,$$

$$\kappa^{ij}(\mathbf{x}) = \kappa_N(x)\left(\delta^{ij} - \frac{x^i x^j}{x^2}\right) + \kappa_L(x)\frac{x^i x^j}{x^2} + g(x)\epsilon^{ijk}x^k, \qquad x = |\mathbf{x}-\mathbf{x}'|,$$

$$\kappa_N(x) = \kappa_L(x) + x\kappa_L'(x)/2 \quad \text{for incompressible velocities.}$$

Equivalently, in the Fourier space we have

$$\langle v^i(\mathbf{k}) v^{*j}(\mathbf{k}) \rangle = F(k)\left(\delta^{ij} - \frac{k^i k^j}{k^2}\right) + iG(k)\,\epsilon^{ijl}k^l.$$

Two independent functions $\kappa_L(x)$ and $g(x)$ or, alternatively, their 3D Fourier transforms $F(k)$ and $G(k)$, describe kinetic energy and helicity.

Homogeneous isotropic magnetic field correlator is

$$\langle B^i(\mathbf{x},t)B^j(0,t)\rangle = M_N(x,t)\left(\delta^{ij} - \frac{x^i x^j}{x^2}\right) + M_L(x,t)\frac{x^i x^j}{x^2} + K(x,t)\,\epsilon^{ijk}x^k,$$

$$M_N(x,t) = M_L(x,t) + (x/2)M_L'(x,t) \quad \text{for solenoidal field.}$$

Equivalently, in Fourier space we have

$$\langle B^i(\mathbf{k},t)B^{*j}(\mathbf{k},t)\rangle = F_B(k,t)\left(\delta^{ij} - \frac{k^i k^j}{k^2}\right) - i\frac{H(k,t)}{2k^2}\epsilon^{ijl}k^l. \tag{1}$$

Here $F_B(k,t)$ is the magnetic energy spectral function, and $H(k,t)$ is the spectral function of the electric current helicity.

- Given velocities correlator, the goal is to find $M_L(x,t)$ and $K(x,t)$ or, alternatively, their Fourier transforms $F_B(k,t)$ and $H(k,t)$.

- When the velocity field is helical, one needs to solve 2 coupled partial differential equations for magnetic field correlator functions (Vainshtein & Kichatinov 1986).
- Self-adjoint form of these equations was found by Boldyrev *et al.* (2005):
  - a quantum mechanical "spinor" form with imaginary time,

$$-\partial_t \psi^\alpha = \hat{\mathcal{H}}^{\alpha\beta}\psi^\beta, \quad \alpha = \{1, 2\};$$

  - two components of $\psi^\alpha(x, t)$ are related to two functions in the magnetic correlator $\langle B^i(\mathbf{x}, t) B^j(0, t)\rangle$;
  - self-adjoint Hamiltonian $\hat{\mathcal{H}}^{\alpha\beta}$ depends on kinetic energy and kinetic helicity, and on magnetic diffusivity η;
  - eigenvalues of $\hat{\mathcal{H}}^{\alpha\beta}$ must be real because it is self-adjoint;
  - magnetic field growth rates λ correspond to negative eigenvalues.
- These equations were solved numerically by using the 4^{th}-order Runge-Kutta numerical integration method.
- The growth rates λ and spectra of the magnetic field eigenmodes were found.

Results for Kolmogorov Turbulence

Consider velocity fluctuations with Kolmogorov power spectrum.

$$F(k) = k^{-13/3}, \qquad 2 \le k \le k_{\max}, \quad -1 \le h \le 1,$$

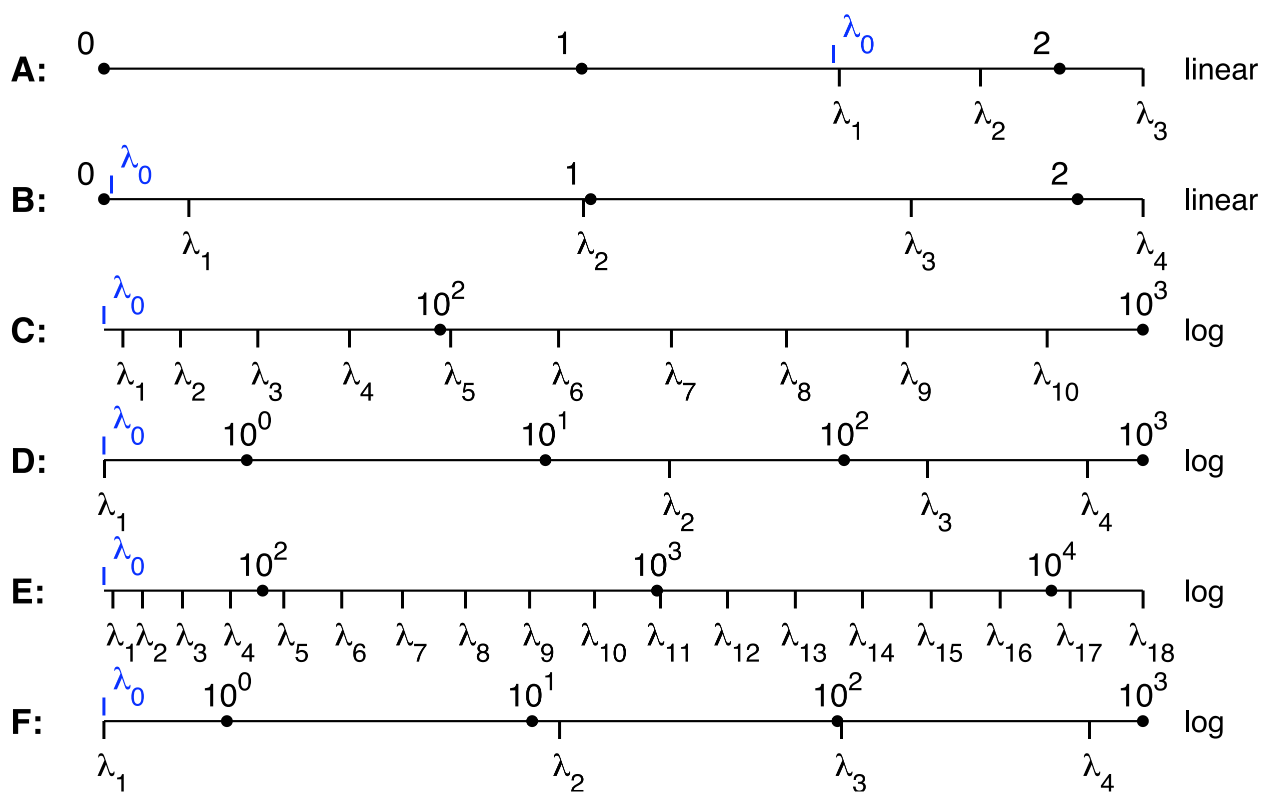
$$G(k) = -hk^{-1}F(k), \qquad k_{\max} \approx 2[\kappa_L(0)/\nu]^{3/4} \approx 4\nu^{-3/4}$$

Growth rates λ_n of field eigenmodes, $\eta = 10^{-6}$, λ_0 is the fastest unbound mode

A & B: $h = 1$ & 0.1,
$k_{\max} = 3$,
$\mathrm{Re} \sim 1$,
$\mathrm{Pr} \gg 1$.

C & D: $h = 1$ & 0.1,
$k_{\max} = 3000$,
$\mathrm{Re} \gg 1$,
$\mathrm{Pr} \gg 1$.

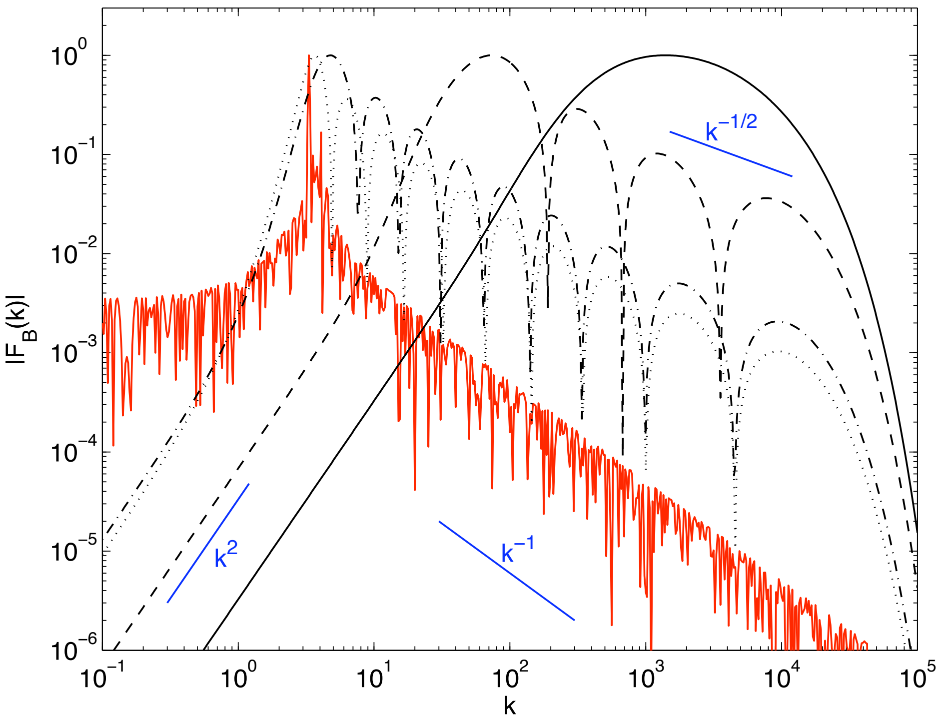
E & F: $h = 1$ & 0.1,
$k_{\max} = 3\times 10^7$,
$\mathrm{Re} \gg 1$,
$\mathrm{Pr} \ll 1$.



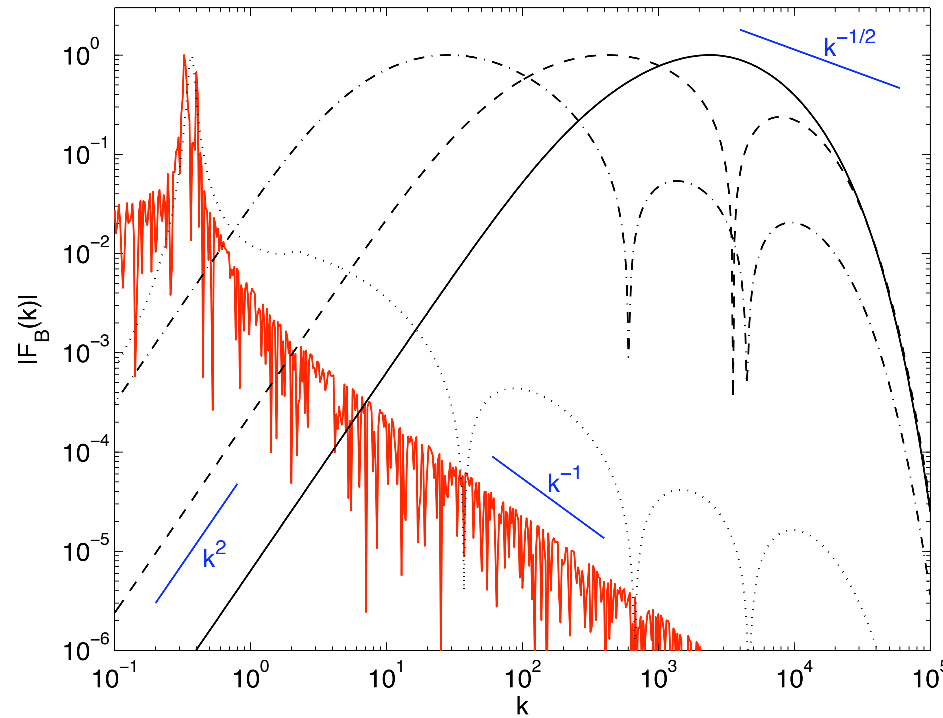
Plots of absolute value of the magnetic field spectral function $F_B(k)$

$\eta = 10^{-6}$, velocities with Kolmogorov spectrum up to $k_{\max} = 3000$ (Re $\gg 1$, Pr $\gg 1$)

Maximal helicity $h = 1$. Red line is for fastest growing unbound eigenmode $\lambda_0 = 33.23$. Solid, dashed, dash-dot, dotted lines are for bound modes $\lambda_{10} = 731.1 > \lambda_7 = 213.4 > \lambda_2 = 42.74 > \lambda_1 = 35.41$.



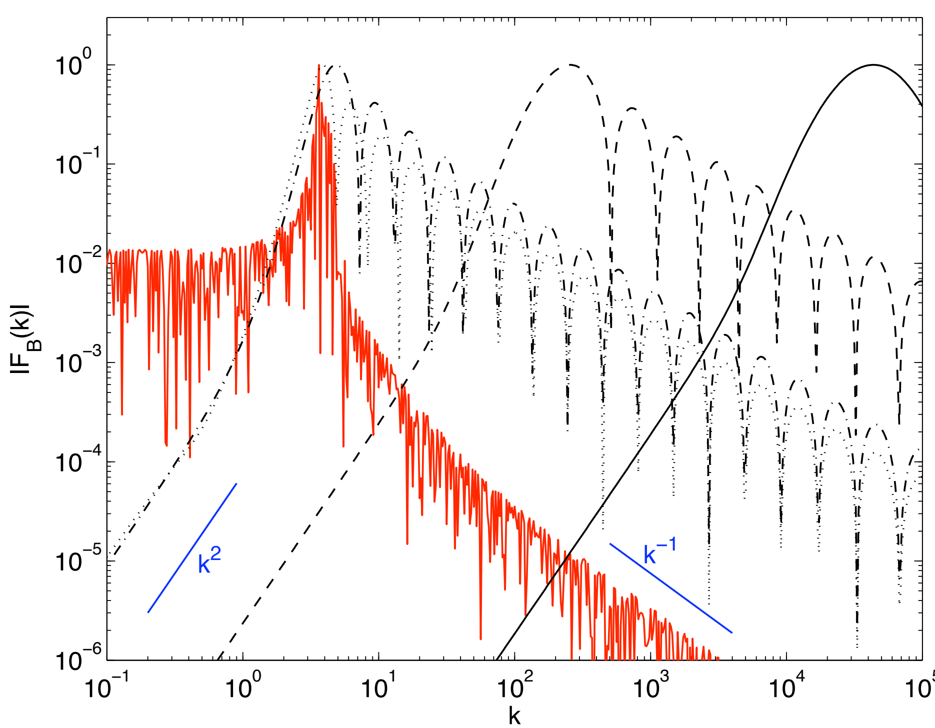
Small helicity $h = 0.1$. Red line is for unbound eigenmode $\lambda_0 = 0.3323$. Solid, dashed, dash-dot, dotted lines are for bound eigenmodes $\lambda_4 = 654.2 > \lambda_3 = 190.1 > \lambda_2 = 26.03 > \lambda_1 = 0.3336$.



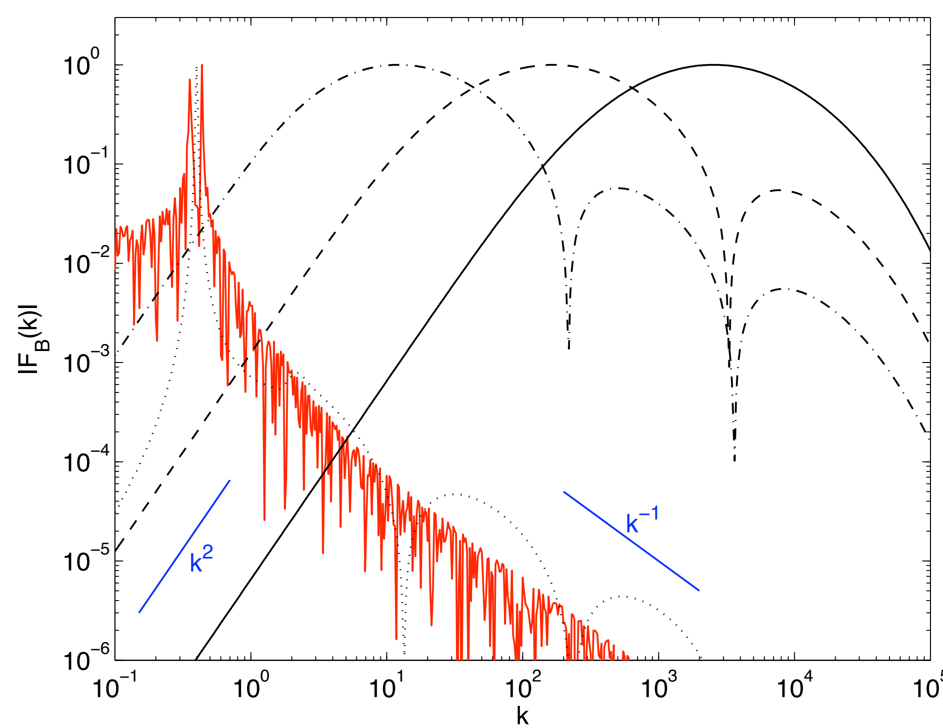
Plots of absolute value of the magnetic field spectral function $F_B(k)$

$\eta = 10^{-6}$, velocities with Kolmogorov spectrum up to $k_{\max} = 3\times10^7$ (Re $\gg 1$, Pr $\ll 1$)

Maximal helicity $h = 1$. Red line is for fastest growing unbound eigenmode $\lambda_0 = 39.57$. Solid, dashed, dash-dot, dotted lines are for bound modes $\lambda_{18} = 17055 > \lambda_{10} = 695.9 > \lambda_2 = 49.56 > \lambda_1 = 41.73$.



Small helicity $h = 0.1$. Red line is for unbound eigenmode $\lambda_0 = 0.39573$. Solid, dashed, dash-dot, dotted lines are for bound modes $\lambda_4 = 669.8 > \lambda_3 = 103.5 > \lambda_2 = 12.30 > \lambda_1 = 0.39581$.



Results and Conclusions

- For growing field eigenmodes it was found that, in contrast with the nonhelical case, in the helical case:
  - when Re >> 1, there is always a shallow bound eigenmode λ_1 (such that $\lambda_1 - \lambda_0 \ll \lambda_0$), whose scale is >> the velocity correlation scale;
  - both bound and unbound eigenmodes contribute to the large-scale magnetic field growth (even in the kinematic regime).
- Conventional alpha-model has limited applicability for large-scale dynamo. This model does not capture bound eigenmodes, while shallow bound eigenmodes rapidly become dominant over unbound modes during the kinematic dynamo regime.
- http://arxiv.org/abs/0711.0973, http://arxiv.org/abs/0810.2950
- Current work: Compare Kazantsev model predictions to DNS results.

END

The self-adjoint equations for $M_L(x,t)$ and $K(x,t)$ (Boldyrev et al 2005) are

$$M_L = \frac{\sqrt{2}e^{\lambda t}}{x^2} w_2(x), \quad K = -\frac{e^{\lambda t}}{\sqrt{2}x^4}\left[x^2 w_3(x)\right]',$$

$$\begin{bmatrix} -\frac{\sqrt{2}}{x}\hat{E}\frac{\sqrt{2}}{x} - \lambda & \frac{\sqrt{2}}{x^3}C\frac{d}{dx}x^2 \\ -x^2\frac{d}{dx}C\frac{\sqrt{2}}{x^3} & x^2\frac{d}{dx}\frac{B}{x^4}\frac{d}{dx}x^2 - \lambda \end{bmatrix} \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} = 0,$$

$$\begin{aligned} \hat{E} &= -\tfrac{1}{2}x\tfrac{d}{dx}B\tfrac{d}{dx}x + \tfrac{1}{\sqrt{2}}(A - xA'), \\ A(x) &= \sqrt{2}\left[2\eta + \kappa_N(0) - \kappa_N(x)\right], \\ B(x) &= 2\eta + \kappa_L(0) - \kappa_L(x), \\ C(x) &= \sqrt{2}\left[g(0) - g(x)\right]x. \end{aligned}$$

- The above equations describe the magnetic field growth in the Kazantsev-Kraichnan model. These equations were numerically solved by using the 4^{th}-order Runge-Kutta numerical integration method.
- The growth rates λ and spectra of the magnetic field eigenmodes were found (note that all λ-s must be real because equations are self-adjoint).

Results for Gaussian Turbulent Velocity Field

Consider Gaussian functions of the velocity correlation tensor (all velocity fluctuations are on scale of order unity, Re $\sim$ 1),

$$\kappa_L(x) = e^{-x^2}, \quad g(x) = \frac{4h\sqrt{2e}}{27\sqrt{3}} \left(5 - \frac{4x^2}{3}\right) e^{-2x^2/3}, \quad -1 \leq h \leq 1.$$

Growth rates λ_n of field eigenmodes, λ_0 is the fastest unbound mode

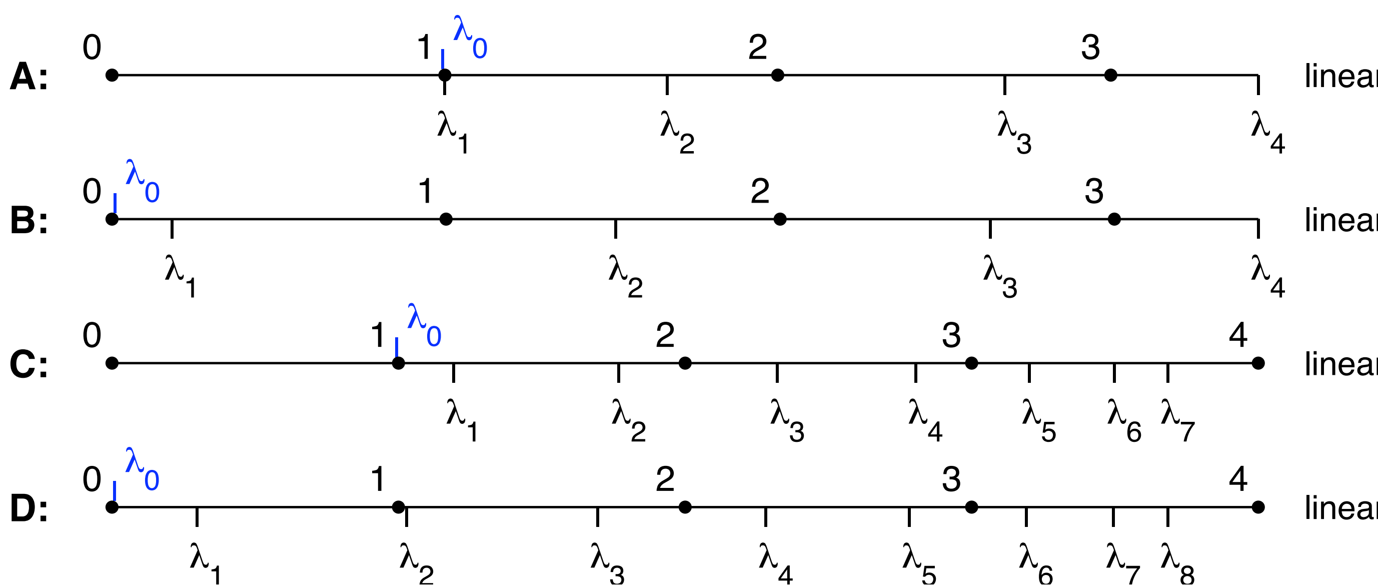
Re ~ 1, Pr ~ 1

A: $\eta = 10^{-6}$, $h = 1.$

B: $\eta = 10^{-6}$, $h = 0.01.$

C: $\eta = 10^{-12}$, $h = 1.$

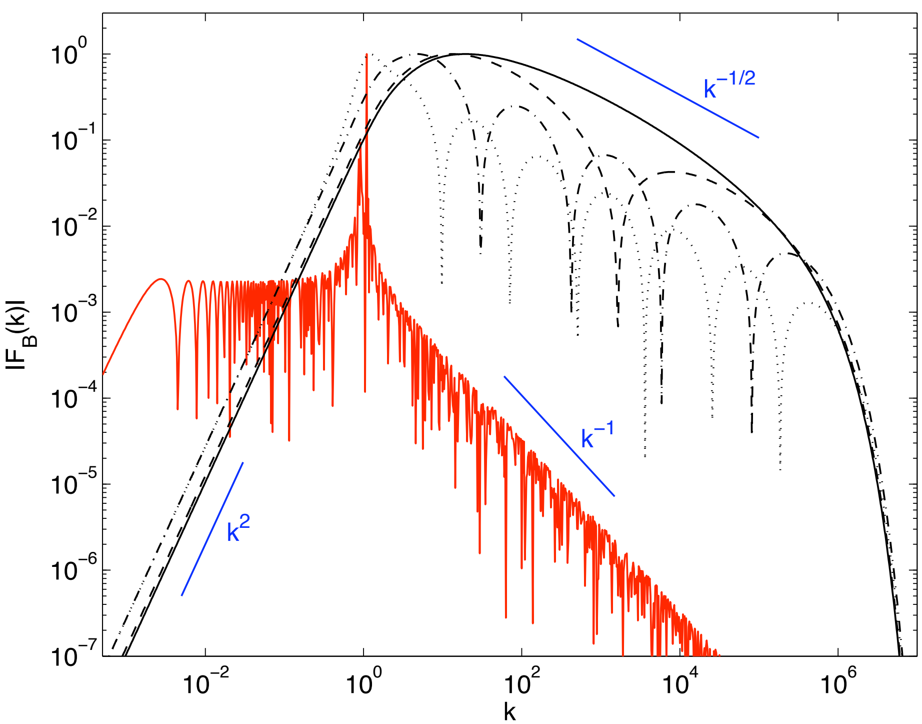
D: $\eta = 10^{-12}$, $h = 0.01.$



Plots of absolute value of the magnetic field spectral function $F_B(k)$

$\eta = 10^{-12}$ and single-scale Gaussian velocities (Re ~ 1, Pr $\gg 1$)

Maximal helicity $h = 1$. Red line is for fastest growing unbound eigenmode $\lambda_0 = 0.9943$. Solid, dashed, dash-dot, dotted lines are for bound modes $\lambda_7 = 3.686 > \lambda_6 = 3.499 > \lambda_3 = 2.322 > \lambda_1 = 1.192$.



Small helicity $h = 0.01$. Red line is for unbound eigenmode $\lambda_0 = 9.943 \times 10^{-5}$. Solid, dashed, dash-dot, dotted lines are for bound eigenmodes $\lambda_8 = 3.685 > \lambda_7 = 3.494 > \lambda_3 = 1.694 > \lambda_1 = 0.2948$.